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Two-extended Toda fields in three dimensions

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Abstract. Two-extended Toda fields associated with Saveliev–Vershik's continuum Lie algebras are studied. Such fields satisfy three-dimensional (integro-)differential equations. Structures such as the fundamental Poisson relation, classical Yang–Baxter equation, chiral exchange algebra and dressing transformations are recovered, which are in complete analogy to the two-dimensional case. The formal solution is also considered using Leznov–Saveliev analysis, but this does not lead to explicit solutions because we have insufficient knowledge about the highest-weight representations of the continuum Lie algebras.

1. Introduction

Two-dimensional integrable field theories have proved to be very fruitful over the last twenty years. Among such models, Toda field theories have received particular attentions because they are related to most of the important subjects of modern theoretical physics. For example, conformal Toda models have played an important role in the investigation of extended conformal algebras (W algebras) and W gravities; their affine and conformal affine analogues were shown to be interesting models as solitonic equations, and also as prototypes of critical–off-critical conformal field theories. Also, the quantum Toda field theories are among the important quantum integrable field theories which admit the beautiful quantum group symmetries and/or factorizable S-matrices. Besides, the study of Toda field theories really helps to establish and understand the systematic methods for treating two-dimensional integrable systems.

Recently, accompanying the investigation of the so-called W-infinity (W_∞, $W_{1+\infty}$ and w_∞) algebras, there arose a wide interest in studying a certain kind of three-dimensional integrable model, especially the well known KP hierarchy and the 'continuous limit' of Toda theories [1–6]. The latter, being related to w_∞ algebra [1–3] and real Euclidean self-dual Einstein gravity [7] and possessing physically non-trivial instanton solutions [4], has been studied by numerous authors from various view points. However, as far as we know, the study of three-dimensional Toda model has not been put forward to the same extent as in the two-dimensional case. One of the still open question is whether, in the three-dimensional case, there exist structures like the 'fundamental Poisson relation' in the sense of the *St Petersburg* (formerly Leningrad) group [8], or whether one can treat the three-dimensional integrable models using the Hamiltonian techniques developed for two-dimensional integrable models.

It seems to us that it may be too ambitious to answer the last question at the present stage because we have not even made clear enough what we mean by the term 'integrability'

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in three-dimensions. In the two-dimensional case, a system of nonlinear partial differential equations is said to be integrable if it can be represented by the 'zero-curvature' equation

$$[\partial_+ - A_+,\ \partial_- - A_-] = 0$$

where the potentials $A_\pm$ are usually Lie algebra valued. This definition of integrability is certainly different from the classical concept of the Liouville integrability. It is sometimes referred to as the Lax integrability because the 'zero-curvature' equation is just the compatibility condition of the Lax pair $\partial_\pm T = A_\pm T$, and it has been proved that the Lax integrability is reduced to the Liouville integrability for two-dimensional systems if and only if the Lax operator $A_1 = \frac{1}{2}(A_+ - A_-)$ possesses a classical r-matrix structure [9]. However, for the three-dimensional case, even Lie algebra valued Lax potentials $A_\pm$ are hard to find for most systems. So, whenever we are speaking of integrable three-dimensional models, we are talking about those models which are solved exactly in some way. Therefore, we seem to be a very long way from establishing a systematic approach for three-dimensional integrable field theories.

Fortunately, due to the development of the concept of contragradient continuum Lie algebras by Saveliev and Vershik [5], one is now able to establish the Lax pair representations for a number of three-dimensional models. The simplest example is just the three-dimensional Toda model mentioned above. Although the continuum Lie algebra valued Lax pair representation for the three-dimensional Toda model looks rather formal at first sight, it seems to us that this is precisely the right way to generalize the theories of two-dimensional integrability to the case of three dimensions. In this article, we shall study three-dimensional generalization of the two-extended Toda model proposed by us some time earlier [10]. We shall try to generalize many of the concepts and methods of two-dimensional integrable models to the three-dimensional case based on this model. As a by-product, we point out that the model under consideration should correspond to a new type of W-infinity algebra, and may probably be denoted by $w_\infty^{(2)}$. The calculation of the explicit structure of this algebra is planned for the future.

2. Review of continuum Lie algebras

Before constructing the three-dimensional two-extended Toda model, let us first give a brief review of the continuum Lie algebras. Due to Saveliev and Vershik [5], the contragradient continuum Lie algebra $\mathcal{G}(E, \mathcal{K}, S)$ is defined as the quotient algebra $\mathcal{G}'(E, \mathcal{K}, S)/J$, where E is a vector space over some field ϕ, $\mathcal{K}$ and S are bilinear mappings $E \times E \to E$, and $\mathcal{G}'$ is the Lie algebra freely generated by the 'local part' $\mathcal{G}' \equiv \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{+1}$ through the relations

$$\begin{aligned}
&[X_0(\varphi),\ X_0(\psi)] = 0\\
&[X_0(\varphi),\ X_{\pm 1}(\psi)] = \pm X_{\pm 1}(\mathcal{K}(\varphi, \psi))\\
&[X_{+1}(\varphi),\ X_{-1}(\psi)] = X_0(S(\varphi, \psi))
\end{aligned}$$

where $\varphi,\ \psi \in E$, and J is the largest homogeneous ideal having a trivial intersection with $\mathcal{G}_0$. For the above relations to really define a Lie-algebra structure, the mappings $\mathcal{K}$ and S have to be subjected to additional constraints. As a special case, one can choose E to be a commutative associative algebra with the multipication $\bullet$, and $\mathcal{K}$ and S to have a *linear* form, say,

$$\mathcal{K}(\varphi, \psi) = \mathcal{K}(\varphi) \bullet \psi \qquad S(\varphi, \psi) = S(\varphi \bullet \psi).$$

In these cases one can further choose $S = \mathrm{id}$, which corresponds to the so-called *standard form* [5]. In this article, we are particularly interested in the standard contragradient continuum Lie algebras with E being the algebra of C^∞ functions on some one-dimensional manifold $\mathcal{M}$ with the local coordinate t. In such cases, one can replace the generating relations by the following relations between the 'kernel generators'

$$[h(t), h(t')] = 0 \qquad [h(t), e_\pm(t')] = \pm K(t,t')e_\pm(t') \qquad [e_+(t), e_-(t)] = \delta(t-t')h(t)$$

where $K(t,t')$ is called the Cartan operator, or, to be exact, the kernel of the Cartan operator $\mathcal{K}$ (we shall use the term *Cartan operator* for both $K(t,t')$ and $\mathcal{K}$, with some abuse of terminology) and

$$X_i(\varphi) \equiv \int \mathrm{d}t\, X_i(t)\varphi(t) \qquad X_0 \equiv h(t) \qquad X_{\pm 1}(t) \equiv e_\pm(t).$$

In general, the Cartan operator may be an integral operator possessing a continuous spectrum (in contrast to the Kac–Moody algebra in which the Cartan matrix possesses a 'discrete spectrum', namely its eigenvalues), and it may or may not be symmetrizable ($K(t,t')$ *is symmetrizable if there exists a function* $v(t)$ *such that* $\mathcal{Q}(t,t') \equiv K(t,t')v(t') = K(t',t)v(t)$. *The operator* $\mathcal{Q}(t,t')$ *is called the symmetrized Cartan operator*). In this article, we shall always assume that the Cartan operator is symmetrizable. As a concrete example, we can choose $K(t,t') = \partial_t^2\delta(t-t')$, which is itself symmetric under $t \leftrightarrow t'$ (and this algebra corresponds to the continuous limit of the Lie algebra A_∞). As we shall see in the following context, the two-extended Toda model corresponding to this last special choice of K is actually a generalization of the three-dimensional Toda model of [1–4, 7]. Other choices of K can also give three-dimensional generalizations of the two-extended Toda model, but the corresponding equations often appear as integro-differential equations.

The general structure theory for the contragradient continuum Lie algebras is not yet established. Nevertheless, it is enough for us to know that the Killing form can be appropriately defined according to concrete choices of K and S, and by definition, the contragradient continuum Lie algebras are naturally Z-graded. In the case of $S = \mathrm{id}$ with a symmetrizable Cartan operator $K(t,t')$, the Killing form can be defined as

$$\langle h(t),\ h(t')\rangle = K(t,t')v(t') = \mathcal{Q}(t,t') \qquad \langle e_+(t),\ e_-(t')\rangle = v(t)\delta(t-t').$$

In particular, if $K(t,t') = \partial_t^2\delta(t-t')$, the function $v(t)$ can be chosen to be the constant 1, and it was shown in [6] that there exist highest-weight representations for the corresponding Lie algebra $\mathcal{G}(C^\infty\mathcal{M}, \mathcal{K}, \mathrm{id})$, with the highest-weight state denoted by $|\tau\rangle$. These materials are all that is needed for our constructions.

3. Two-extended Toda model in three dimensions

With the above mathematical preparation given, let us now go on to the central subject of this article—the two-extended Toda model in three dimensions.

The two-dimensional case of this model is studied by us in a series of papers [10, 11], in which the W-algebra symmetries, fundamental Poisson relation, chiral exchange algebras, general solution and the Wronskian-type special solution in relation to the WZNW reduction and classical W-surfaces are made clear. The crucial difference between the two-extended Toda model and the standard one lies in that, in the standard case, the Lax connections $A_\pm$ take their respective values in the subspaces $\mathcal{G}_0 \oplus \mathcal{G}^{(\pm 1)}$ of the underlying Lie algebra $\mathcal{G}$ (where $\mathcal{G}^{(\pm i)}$ denote the ith graded sector of the Lie algebra $\mathcal{G}$), whilst for the two-extended model, these connections take values in the subspaces $\mathcal{G}_0 \oplus \mathcal{G}^{(\pm 1)} \oplus \mathcal{G}^{(\pm 2)}$. It is precisely this difference that makes the two-dimensional two-extended Toda model have the extended conformal symmetry algebra $W[\mathcal{G}, H, 2]$, in contrast to the standard $W[\mathcal{G}, H, 1]$ symmetry algebra for the usual Toda model (we are using the convention of [11], where the symbol $W[A_N$, principal gradation, $1]$ corresponds to the usual W_{N+1} algebra).

Let us be more concrete. The Lax pair of the two-extended Toda model can be written as

$$\partial_\pm T = A_\pm T \qquad A_\pm = \pm\left[\tfrac{1}{2}\partial_\pm\Phi + \exp\left(\mp\tfrac{1}{2}\mathrm{ad}\Phi\right)\bar{\Psi}_\pm + \exp\left(\mp\tfrac{1}{2}\mathrm{ad}\Phi\right)\mu_\pm\right] \tag{1}$$

where, in the two-dimensional case, the fields Φ take values in the Cartan subalgebra of the Kac–Moody algebra $\mathcal{G}$, $\bar{\Psi}_\pm$ lie in $\mathcal{G}^{(\pm1)}$, and $\mu_\pm$ are constant, regular, representative elements of $\mathcal{G}^{(\pm2)}$, respectively.

Now in order to generalize this model to the three-dimensional case, we require that the fields Φ, $\bar{\Psi}_\pm$ and the constants $\mu_\pm$ be continuum Lie algebra valued. That means that the above quantities can be rewritten in the form

$$\begin{aligned}
\Phi(x_+,\ x_-) &\equiv \int \mathrm{d}t\, h(t)\varphi(x_+,\ x_-,\ t)\\
\Psi_\pm(x_+,\ x_-) &\equiv \int \mathrm{d}t\, e_\mp(t)\psi_\pm(x_+,\ x_-,\ t)\\
\mu_\pm &\equiv \pm\tfrac{1}{2}\int \mathrm{d}t\,\mathrm{d}t'\,\Omega(t,t')[e_\pm(t),\ e_\pm(t')]
\end{aligned} \tag{2}$$

where $\Omega(t,\ t')$ is an antisymmetric function of t and t', and

$$\begin{aligned}
\bar{\Psi}_\pm(x_+,\ x_-) &\equiv \pm[\mu_\pm,\ \Psi_\pm]\\
&= \tfrac{1}{2}\int \mathrm{d}t\,\mathrm{d}t'\,\mathrm{d}t_1\,\Omega(t,t')\psi_\pm(x_+,\ x_-,\ t_1)[[e_\pm(t),\ e_\pm(t')],\ e_\mp(t_1)]\\
&= \int \mathrm{d}t\,\mathrm{d}t'\,\Omega(t,t')K(t,\ t')\psi_\pm(x_+,\ x_-,\ t)e_\pm(t').
\end{aligned} \tag{3}$$

With these definitions in mind, we are now ready to write the equations of motion for the two-extended Toda model. This can be done by first calculating the compatibility condition of the Lax pair, which gives the result

$$\begin{aligned}
&\partial_+\partial_-\Phi + [\exp(\mathrm{ad}\Phi)\bar{\Psi}_-,\ \bar{\Psi}_+] + [\exp(\mathrm{ad}\Phi)\mu_-,\ \mu_+] = 0\\
&\partial_+\Psi_- = \exp(-\mathrm{ad}\Phi)\bar{\Psi}_+\\
&\partial_-\Psi_+ = \exp(\mathrm{ad}\Phi)\bar{\Psi}_-
\end{aligned} \tag{4}$$

and then substituting the definitions (2), (3) into (4). It finally follows that

$$\begin{aligned}
&\partial_+\partial_-\varphi(x_+,x_-,t)\\
&\qquad - \int \mathrm{d}t_1\,\mathrm{d}t_2\,\Omega(t_1,t)\Omega(t_2,t)K(t_1,t)K(t_2,t)\psi_-(x_+,x_-,t_1)\psi_+(x_+,x_-,t_2)\Xi(t)\\
&\qquad + \int \mathrm{d}t_1\,\Omega^2(t_1,t)K(t_1,t)\Xi(t_1)\Xi(t) = 0\\
&\partial_+\psi_-(x_+,x_-,t) = \int \mathrm{d}t_1\,\Omega(t_1,t)\psi_+(x_+,x_-,t_1)K(t_1,t)\Xi(t)\\
&\partial_-\psi_+(x_+,x_-,t) = \int \mathrm{d}t_1\,\Omega(t_1,t)\psi_-(x_+,x_-,t_1)K(t_1,t)\Xi(t)
\end{aligned} \tag{5}$$

where the explicit dependence of Ξ on $x_\pm$ is omitted for the sake of brevity of notations,

$$\Xi(t) = \exp\left(-\int \mathrm{d}\tilde{t}\,K(\tilde{t},t)\varphi(x_+,x_-,\tilde{t})\right). \tag{6}$$

At first glance, the system (5) of integro-differential equations appears to be rather complicated. If, however, the Cartan operator $K(t,t')$ and (correspondingly) the function $\Omega(t,t')$ are chosen appropriately, the three-dimensional two-extended Toda model can be rewritten in a very neat form. For example, if we choose the underlying contragradient continuum Lie algebra to be $\mathcal{G}(C^\infty\mathcal{M},\ \partial_t^2\delta(t-t'),\ \mathrm{id})$ and let $\Omega(t,t')=t-t'$, equation (5) can be rewritten as

$$
\begin{aligned}
&\partial_+\partial_-\varphi - 4\partial_t\psi_-\partial_t\psi_+\exp(-\partial_t^2\varphi) + 2\exp(-2\partial_t^2\varphi) = 0\\
&\partial_+\psi_- = 2\partial_t\psi_+\exp(-\partial_t^2\varphi)\\
&\partial_-\psi_+ = 2\partial_t\psi_-\exp(-\partial_t^2\varphi)
\end{aligned}
\tag{7}
$$

which is exactly an extension of the 'three-dimensional Toda model' studied by several authors.

The reason for the above system being of interest to study is that, firstly, this system of equations is really a continuous limit of the two-dimensional A_∞ two-extended Toda model, just as the usual three-dimensional Toda model,

$$\partial_+\partial_-\varphi + \exp(-\partial_t^2\varphi) = 0$$

is the continuous limit of the two-dimensional A_∞ Toda model. Secondly, as a direct extension of the three-dimensional Toda model, this system is expected to possess many extended characteristics of the three-dimensional Toda model, such as a generalized w_∞ symmetry, extended self-dual Einstein spaces, etc. Actually, the two-extended Toda model with the underlying Lie algebra A_∞ was suggested [10] to possess a conformal symmetry algebra already in the two-dimensional case; it can be denoted $W_\infty^{(2)}$, which is the generalization to the case of the integer–half-integer conformal spectrum of the usual nonlinear W_∞ algebra, or the large-N limit of the $W_N^{(2)}$ algebra. The conformal algebras with integer–half-integer spectra were referred to as the 'bosonic superconformal algebra' by Fuchs [12] and by us. Using this terminology, the algebra $W_\infty^{(2)}$ may be called the 'bosonic super W-infinity' algebra. The $w_\infty^{(2)}$ algebra is supposed to be a linear variant of $W_\infty^{(2)}$, just as w_∞ is a linear variant of W_∞. However, the explicit structure of the algebras $W_\infty^{(2)}$ and $w_\infty^{(2)}$ is still difficult to construct. The central difficulty is that, in the two-extended case, it is not as easy as in the usual case to choose a complete set of chiral conserved quantities as a good basis. Nevertheless, there should be no problem connected with the existence of such bosonic superconformal algebras. We hope the difficulty in choosing a basis for such algebras could be overcome in the future.

It might be of interest to note that the system (7) really admits physically interesting solutions. For example, the instanton-like solution

$$
\begin{aligned}
&\varphi = \int^t \mathrm{d}t_1 \int^{t_1} \mathrm{d}t_2 \ln\left[\tfrac{1}{4}(t_2-a)(t_2-b)\frac{\partial_+ f_+(x_+)\partial_- f_-(x_-)}{(1-f_+(x_+)f_-(x_-))^2}\right]\\
&\psi_\pm = g_\pm(x_\pm)
\end{aligned}
$$

explicitly solves equation (7), where $f_\pm$ and $g_\pm$ are arbitrary functions of the arguments $x_\pm$. To obtain more solutions of the system, we have to generalize the techniques for solving two-dimensional Toda-type models. However, in the present section, we would rather introduce the effective action for the three-dimensional two-extended Toda model and leave the task of generalizing the techniques for solving two-dimensional Toda-type models to the next section.

The effective action for the system (4), (5) reads [10]

$$I(\Phi,\Psi_\pm)=\frac{1}{4}\int \mathrm{d}x_+\,\mathrm{d}x_-\langle\partial_+\Phi\partial_-\Phi+\bar\Psi_+\partial_-\Psi_++\bar\Psi_-\partial_+\Psi_-\rangle$$
$$-\frac{1}{2}\int \mathrm{d}x_+\,\mathrm{d}x_-\left\langle\exp(-\mathrm{ad}\Phi)(\bar\Psi_+)\bar\Psi_-+\exp(-\mathrm{ad}\Phi)(\mu_+)\mu_-\right\rangle \tag{8}$$
$$=\frac{1}{4}\int \mathrm{d}x_+\,\mathrm{d}x_-\,\mathrm{d}t\,v(t)\,\{\partial_+\varphi(\mathcal{K})(\partial_-\varphi)+(\mathcal{K}\Omega)(\psi_+)\partial_-\psi_++(\mathcal{K}\Omega)(\psi_-)\partial_+\psi_-$$
$$-2(\mathcal{K}\Omega)(\psi_+)(\mathcal{K}\Omega)(\psi_-)\Xi+(\mathcal{K}\Omega^2)(\Xi)\Xi\} \tag{9}$$

where

$$(\mathcal{K})(f)\equiv\int \mathrm{d}t_1\,K(t_1,t)f(t_1)$$
$$(\mathcal{K}\Omega)(f)\equiv\int \mathrm{d}t_1\,K(t_1,t)\Omega(t_1,t)f(t_1)$$
$$(\mathcal{K}\Omega^2)(f)\equiv\int \mathrm{d}t_1\,K(t_1,t)\Omega^2(t_1,t)f(t_1).$$

In the particular case of the $\mathcal{G}(C^\infty\mathcal{M},\ \partial_t^2\delta(t-t'),\ \mathrm{id})$ model with $\Omega(t,t')=t-t'$, the above action can be rewritten as

$$I(\varphi,\psi_\pm)=\frac{1}{4}\int \mathrm{d}x_+\,\mathrm{d}x_-\left\{\partial_+\varphi\partial_-\partial_t^2\varphi+2\partial_t\psi_+\partial_-\psi_++2\partial_t\psi_-\partial_+\psi_-\right.$$
$$\left.-8\partial_t\psi_+\partial_t\psi_-\exp(-\partial_t^2\varphi)+2\exp(-2\partial_t^2\varphi)\right\}. \tag{10}$$

Note that the last action is of fourth-order in derivatives of field φ. Such an action does not describe a three-dimensional relativistic field theory in the usual sense. Actually, the extra dimension t is an algebraical dimension which has different meaning in contrast to the other two space-time dimensions.

Since we are not experienced in treating fourth-order actions, we now prefer to define the canonical Poisson brackets for the original form (8) of the action. Remembering that the fields Φ, $\Psi_\pm$ are just continuum Lie algebra valued *two-dimensional* fields, and the action (8) for these fields is of second order, we can define the Poisson bracket for these fields in the usual way. That means that we can define the canonical conjugate momenta Π_Φ, $\Pi_{\Psi_\pm}$ as

$$\Pi_\Phi\equiv\frac{\delta\mathcal{L}}{\delta\partial_0\Phi}=\frac{1}{2}\partial_0\Phi\qquad \Pi_{\Psi_\pm}\equiv\frac{\delta\mathcal{L}}{\delta\partial_0\Psi_\pm}=\frac{1}{2}\bar\Psi_\pm \tag{11}$$

where x_0 and x_1 are defined as $x_\pm=x_0\pm x_1$, and introduce the canonical equal-time x_0 Poisson brackets

$$\begin{aligned}\left\{\Pi_\Phi(x_1)\otimes,\ \Phi(x_1')\right\}&=\delta(x_1-x_1')\int \mathrm{d}t\,\mathrm{d}t'\mathcal{Q}^{-1}(t,t')h(t)\otimes h(t')\\ \left\{\Pi_{\Psi_\pm}(x_1)\otimes,\ \Psi_\pm(x_1')\right\}&=\delta(x_1-x_1')\int \mathrm{d}t\,v^{-1}(t)e_\pm(t)\otimes e_\mp(t)\end{aligned} \tag{12}$$

where $\mathcal{Q}^{-1}(t,t')$ is the formal inverse of the symmetrized Cartan operator $\mathcal{Q}(t,t')$,

$$\int \mathrm{d}t\,\mathcal{Q}^{-1}(t_1,t)\mathcal{Q}(t,t_2)=\int \mathrm{d}t\,\mathcal{Q}(t_1,t)\mathcal{Q}^{-1}(t,t_2)=\delta(t_1-t_2).$$

Remembering the definitions (11) of the fields Π_Φ, $\Pi_{\Psi_\pm}$ and requiring the Poisson bracket for the component fields φ, $\psi_\pm$ to be consistent with those defined above, we find that the

Poisson bracket for the fields φ, $\psi_\pm$ may be appropriately defined as

$$\begin{aligned}&\left\{\tfrac{1}{2}(\mathcal{Q})(\partial_0\varphi)(x_1,t),\varphi(x_1',t')\right\}=\delta(x_1-x_1')\delta(t-t')\\&\left\{\tfrac{1}{2}(\mathcal{K}\Omega)(\psi_\pm)(x_1,t),\psi_\pm(x_1',t')\right\}=\delta(x_1-x_1')\delta(t-t').\end{aligned}\tag{13}$$

These Poisson brackets are exactly what will arise if we treat directly the action (9) and define the Poisson brackets in the usual way. Therefore, in spite of the different meaning of the variable t, the Poisson brackets can be defined safely using the usual method.

4. Fundamental Poisson relation and classical Yang–Baxter equation

As mentioned in the last section, the Hamiltonian method of treating higher-dimensional ($d > 2$) integrable systems is still lacking, such as the fundamental Poisson structure and classical r-matrices. For some particular examples, such as the self-dual Yang–Mills theory (where the Lax-type linear systems were found for a long time by the use of the prolongation method [13]), the classical r-matrix has been found by Chau *et al* [14] using the J-field formulation. However, up to now, there is very little knowledge about whether there may be a fundamental Poisson relation for the transport matrices since the transport matrix depends on all four space-time variables, and the Poisson bracket for such matrices cannot be obtained easily by integrating those for the potentials in the linear systems.

Fortunately, in the three-dimensional Toda-type models, the 'transport operator' T does not depend on the third variable t, and thus we can get the Poisson bracket for T by directly integrating the Poisson bracket for the Lax connections.

Let us first calculate the Poisson bracket $\{A_1(x_1)\otimes,\ A_1(x_1')\}$ with $A_1\equiv\frac{1}{2}(A_+-A_-)$. By direct calculation, we have

$$\begin{aligned}\{A_1(x_1)\otimes,\ A_1(x_1')\}=\tfrac{1}{2}\int &\mathrm{d}t\,\mathrm{d}t'\,K(t,t')\Omega(t,t')\Xi^{1/2}(t')\\&\times\left\{\psi_+(t)\left[e_+(t')\otimes h(t')-h(t')\otimes e_+(t')\right]\right.\\&\left.+\,\psi_-(t)\left[e_+(t')\otimes h(t')-h(t')\otimes e_+(t')\right]\right\}\delta(x_1-x_1')\\&+\tfrac{1}{2}\int\mathrm{d}t\,\mathrm{d}t'\,\Omega(t,t')\Xi^{1/2}(t)\Xi^{1/2}(t')\\&\times\left\{[e_+(t),e_+(t')]\otimes\left(h(t)+h(t')\right)-\left(h(t)+h(t')\right)\otimes[e_+(t),e_+(t')]\right.\\&-[e_-(t),e_-(t')]\otimes\left(h(t)+h(t')\right)+\left(h(t)+h(t')\right)\otimes[e_-(t),e_-(t')]\\&+2K(t,t')\left[e_+(t)\otimes e_+(t')-e_+(t')\otimes e_+(t)\right.\\&\left.\left.e_-(t)\otimes e_-(t')-e_-(t')\otimes e_-(t)\right]\right\}\delta(x_1-x_1').\end{aligned}\tag{14}$$

Equation (14) can be rewritten as

$$\{A_1(x_1)\otimes,\ A_1(x_1')\}=[r,A_1(x_1)\otimes 1+1\otimes A_1(x_1')]\delta(x_1-x_1')\tag{15}$$

where r is a $\mathcal{G}\otimes\mathcal{G}$-valued constant, which may be called the 'r-operator',

$$\begin{aligned}r=\tfrac{1}{2}\sum_a\sum_{n=1}^{\infty}\int\mathrm{d}t_1\ldots\int\mathrm{d}t_n&\left\{e_+^{(a)}(t_1,\ \ldots,\ t_n)\otimes e_-^{(a)}(t_1,\ \ldots,\ t_n)\right.\\&\left.-e_-^{(a)}(t_1,\ \ldots,\ t_n)\otimes e_+^{(a)}(t_1,\ \ldots,\ t_n)\right\}+\lambda C\qquad(\lambda\text{ arbitrary})\end{aligned}\tag{16}$$

where $e_\pm^{(a)}(t_1,\ \ldots,\ t_n)\in\mathcal{G}^{(\pm n)}$, the superscript $^{(a)}$ indicates different (linearly independent) elements of $\mathcal{G}^{(\pm n)}$, which are assumed to be normalized such that

$$(e_\pm^{(a)}(t_1,\ \ldots,\ t_n),\ e_\mp^{(b)}(t_1',\ \ldots,\ t_n'))=\delta^{ab}\delta(t_1-t_1')\ldots\delta(t_n-t_n')$$

and the summation over a is taken over all such elements. The constant C is the 'tensor Casimir operator' of the continuum Lie algebra $\mathcal{G}$ satisfying the conditions

$$[C, \mathcal{A} \otimes 1 + 1 \otimes \mathcal{A}] = 0$$
$$\langle C,\ \mathcal{A} \otimes 1\rangle = \langle C,\ 1 \otimes \mathcal{A}\rangle = \mathcal{A} \qquad \forall \mathcal{A} \in \mathcal{G}.$$

Explicitly, we can write

$$C = \int \mathrm{d}t\, \mathrm{d}t'\, Q^{-1}(t, t')h(t) \otimes h(t') + \sum_a \sum_{n=1}^{\infty} \int \mathrm{d}t_1 \ldots \int \mathrm{d}t_n \left\{ e_+^{(a)}(t_1,\ \ldots,\ t_n) \otimes e_-^{(a)}(t_1,\ \ldots,\ t_n) + e_-^{(a)}(t_1,\ \ldots,\ t_n) \otimes e_+^{(a)}(t_1,\ \ldots,\ t_n) \right\}.$$

Of all the choices of λ, two special values $\lambda = \pm\frac{1}{2}$ are particularly important because the corresponding r-operators,

$$r_\pm = \pm\frac{1}{2}\left\{ \int \mathrm{d}t\, \mathrm{d}t'\, Q^{-1}(t, t')h(t) \otimes h(t') + 2\sum_a \sum_{n=1}^{\infty} \int \mathrm{d}t_1 \ldots \int \mathrm{d}t_n\, e_\pm^{(a)}(t_1,\ \ldots,\ t_n) \otimes e_\mp^{(a)}(t_1,\ \ldots,\ t_n) \right\} \tag{17}$$

satisfy the classical Yang–Baxter equation

$$[r_\pm^{12}, r_\pm^{13}] + [r_\pm^{12}, r_\pm^{23}] + [r_\pm^{32}, r_\pm^{13}] = 0.$$

The fundamental Poisson relation for T is then easily obtained by integrating the equation $\partial_1 T = A_1 T$, yielding

$$\{T(x_1)\otimes, T(x_1)\} = [r_\pm, T(x_1) \otimes T(x_1)]. \tag{18}$$

5. Exchange algebra and dressing transformation

Let us now proceed by analogy to the two-dimensional case. In what follows, we assume that the highest-weight representations for $\mathcal{G}$ exist, and the highest-weight vector $|\tau\rangle$ satisfies the equalities

$$h(t)|\tau\rangle = \tau(t)|\tau\rangle \qquad e_+(t)|\tau\rangle = 0 \qquad \forall t$$
$$\langle\tau|\tau\rangle = 1.$$

Note that the Lax pair (1) admits a gauge freedom $T \to gT$ with $g \in G$, the underlying continuum Lie group. Using this gauge degree of freedom, we can transform the Lax connections $A\pm$ so that one of them takes the form $A_\pm \in \mathcal{G}^{(\pm1)} \oplus \mathcal{G}^{(\pm2)}$ or $A_\pm \in \oplus\mathcal{G}^{(\pm2)}$. Then projecting the resulting transformed Lax pairs onto the states $|\tau\rangle$, $\int \mathrm{d}t\, e_-(t)|\tau\rangle$ and their dual states, we can get two sets of 'chiral' vectors which means that they depend on only one space-time variable, x_+ or x_-.

To be explicit, we have the chiral vectors,

$$\xi_1^+(x_+) = \langle\tau| \exp(\tfrac{1}{2}\Phi)T \qquad \xi_1^-(x_-) = T^{-1}\exp(\tfrac{1}{2}\Phi)|\tau\rangle$$
$$\xi_2^+(x_+) = \left\{\langle\tau| \int \mathrm{d}t\, e_+(t)\right\} \exp(\Psi_+)\exp(\tfrac{1}{2}\Phi)T \tag{19}$$
$$\xi_2^-(x_-) = T^{-1}\exp(\tfrac{1}{2}\Phi)\exp(\Psi_-)\left\{\int \mathrm{d}t\, e_-(t)|\tau\rangle\right\}$$

with

$$\partial_\pm \xi_a^\mp = 0 \qquad a = 1,\ 2.$$

Following the standard method [15], we can show that these chiral vectors satisfy the exchange algebra [10]

$$\begin{aligned}
\left\{\xi_a^+(x)\otimes,\ \xi_b^+(y)\right\} &= \xi_a^+(x)\otimes\xi_b^+(y)\,(r_+\theta(x-y)+r_-\theta(y-x))\\
\left\{\xi_a^+(x)\otimes,\ \xi_b^-(y)\right\} &= -\left(\xi_a^+(x)\otimes 1\right) r_-\left(1\otimes\xi_b^-(y)\right)\\
\left\{\xi_a^-(x)\otimes,\ \xi_b^+(y)\right\} &= -\left(1\otimes\xi_b^-(y)\right) r_+\left(\xi_a^+(x)\otimes 1\right)\\
\left\{\xi_a^-(x)\otimes,\ \xi_b^-(y)\right\} &= (r_-\theta(x-y)+r_+\theta(y-x))\,\xi_a^-(x)\otimes\xi_b^-(y).
\end{aligned} \tag{20}$$

Let us now consider the dressing problem of the three-dimensional two-extended Toda model. As in the two-dimensional case, the dressing transformation depends essentially on the factorization of the underlying Lie group G, under which each group element g is factorized as

$$g = g_-^{-1} g_+ \tag{21}$$

and the dressing transformation transforms the transport operator T as

$$T \to T^g = \Theta_\pm T g_\pm^{-1} \qquad \Theta_-^{-1}\Theta_+ = \Theta \equiv T g T^{-1}. \tag{22}$$

At present, the factorization problem is solved by the r-operators $r_\pm$ as follows,

$$\mathcal{A}_\pm \equiv \mathcal{R}_\pm \mathcal{A} \equiv \langle r_\pm,\ 1\otimes\mathcal{A}\rangle_2 \qquad \Rightarrow \mathcal{A} = \mathcal{A}_+ - \mathcal{A}_- \tag{23}$$

which is the infinitesimal form of the factorization problem. The fact that the positive and negative transformations of T give rise to the same T^g implies that the transformed Lax potentials $A_\pm^g$ have the same form as that of the original $A_\pm$. Recalling the concrete form (17) of the $r_\pm$-operators, we can rewrite the $\Theta_\pm$ operators in the following Z-graded form,

$$\Theta_\pm = \exp\left(\tfrac{1}{2}\theta_\pm^{(0)}\right)\exp(\theta^{(\pm 1)})\ldots\exp(\theta^{(\pm 1)})\ldots \tag{24}$$

where $\theta^{(a)} \in \mathcal{G}^{(a)}$, and the form-preserving condition for the Lax connections $A_\pm$ implies that the fields Φ, $\Psi_\pm$ must transform as [10]:

$$\begin{aligned}
&\Phi^g = \Phi + \theta_+^{(0)} = \Phi - \theta_-^{(0)} \qquad \theta_+^{(0)} + \theta_-^{(0)} = 0\\
&\Psi_\pm^g = \Psi_\pm \mp \exp\left(\pm\tfrac{1}{2}\mathrm{ad}\Phi\right)\theta^{(\mp 1)}.
\end{aligned}$$

Note that each element $\theta^{(0)} \in \mathcal{G}^{(0)}$ can be written as $\int \mathrm{d}t\,\theta^{(0)}(t)h(t)$, and each $\theta^{(\pm 1)} \in \mathcal{G}^{(\pm 1)}$ can be written as $\int \mathrm{d}t\,\theta^{(\pm 1)}(t)e_\pm(t)$, so that we can rewrite the above dressing transformation laws in terms of the component fields,

$$\begin{aligned}
\varphi^g(x_+, x_-, t) &= \varphi(x_+, x_-, t) \pm \theta_\pm^{(0)}(x_+, x_-, t)\\
\psi_\pm^g(x_+, x_-, t) &= \psi_\pm(x_+, x_-, t) \mp \Xi^{1/2}(t)\theta^{(\mp 1)}(x_+, x_-, t).
\end{aligned}$$

It is particularly interesting to note that the chiral vectors $\xi_a^\pm$ transform only by a shift of constant group elements,

$$(\xi_a^+)^g = \xi_a^+ g_-^{-1} \qquad (\xi_a^-)^g = g_+\xi_a^-.$$

In order that the above transformations preserve the form of the chiral exchange algebra (20), the constant group elements $g_\pm$ must be subject to some non-trivial Poisson brackets,

$$\begin{aligned}
\{g_+\otimes, g_+\} &= [r_\pm,\ g_+\otimes g_+]\\
\{g_-\otimes, g_-\} &= [r_\pm,\ g_-\otimes g_-]\\
\{g_+\otimes, g_-\} &= [r_+,\ g_+\otimes g_-]\\
\{g_-\otimes, g_+\} &= [r_-,\ g_-\otimes g_+].
\end{aligned}$$

These structures, while considered in the framework of two-dimensional Toda-type theories, correspond to the semiclassical limit of the quantum (Kac–Moody) group [16]. Therefore, it might be interesting to see whether the above structure can give rise to any structure like a 'quantum continuum Lie group' after quantization. We leave this problem open for later considerations.

6. Formal solutions via Leznov–Saveliev analysis [17]

As in the two-dimensional case, the chiral vectors $\xi_a^{\pm}$ are also useful for constructing formal solutions of the two-extended Toda system (4), (5). Recalling the definitions of these chiral vectors, we have

$$\begin{aligned}
&\langle\tau|\exp(\Phi)|\tau\rangle = \xi_1^+(x_+)\xi_1^-(x_-)\\
&\langle\tau|\exp(\Phi)\exp(\Psi_-)\left\{\int \mathrm{d}t\, e_-(t)|\tau\rangle\right\} = \xi_1^+(x_+)\xi_2^-(x_-)\\
&\left\{\int \mathrm{d}t\langle\tau|e_+(t)\right\}\exp(\Psi_+)\exp(\Phi)|\tau\rangle = \xi_2^+(x_+)\xi_1^-(x_-).
\end{aligned}$$

In terms of the component fields, the above equations read

$$\begin{aligned}
&\exp\int \mathrm{d}t\,\varphi(x_+,x_-,t)\tau(t) = \xi_1^+(x_+)\xi_1^-(x_-)\\
&\int \mathrm{d}t\,\psi_-(x_+,x_-,t)\tau(t) = \frac{\xi_1^+(x_+)\xi_2^-(x_-)}{\xi_1^+(x_+)\xi_1^-(x_-)}\\
&\int \mathrm{d}t\,\psi_+(x_+,x_-,t)\tau(t) = \frac{\xi_2^+(x_+)\xi_1^-(x_-)}{\xi_1^+(x_+)\xi_1^-(x_-)}.
\end{aligned}\tag{25}$$

Let us consider the above relations in more detail and show how we can obtain formal solutions from these relations.

Define $T_{\mathrm{L/R}} = \exp(\pm\frac{1}{2}\Phi)T$ and decompose them as

$$T_{\mathrm{L}} = \mathrm{e}^{K_+}N_-M_+ \qquad T_{\mathrm{R}} = \mathrm{e}^{K_-}N_+M_-$$

where $K_\pm \in G_0$, $N_\pm$, $M_\pm \in G_\pm$, we can get from equation (19) that

$$\xi_1^+(x_+) = \langle\tau|\mathrm{e}^{K_+}M_+ \qquad \xi_1^-(x_-) = M_-^{-1}\mathrm{e}^{-K_-}|\tau\rangle$$

which shows that $K_\pm$ and $M_\pm$ are chiral objects,

$$\partial_\pm K_\mp = \partial_\pm M_\mp = 0.$$

Furthermore, writing $N_\pm = \exp(\chi^{(\pm1)})\exp(\chi^{(\pm2)})\ldots$ as we did in equation (24), we obtain [10]

$$\begin{aligned}
&\xi_2^+(x_+) = \left\{\int \mathrm{d}t\langle\tau|e_+(t)\right\}\mathrm{e}^{K_+}\left[1+\mathrm{e}^{-\mathrm{ad}K_+}P_+\right]M_+\\
&\xi_2^-(x_-) = M_-^{-1}\left[1+\mathrm{e}^{-\mathrm{ad}K_-}P_-\right]\mathrm{e}^{-K_-}\left\{\int \mathrm{d}t\, e_-(t)|\tau\rangle\right\}
\end{aligned}$$

where

$$P_\pm \equiv \Psi_\pm \pm \exp(\mathrm{ad}K_\pm)\chi^{(\mp1)} \qquad \partial_\pm P_\mp = 0.$$

More detailed calculations [10] show that the operators $M_\pm$ are not independent of $K_\pm$ and $P\pm$,

$$\partial_+ M_+ M_+^{-1} = \exp(-\mathrm{ad}K_+)(\bar{P}_+ + \mu_+)$$
$$M_- \partial_- M_-^{-1} = \exp(-\mathrm{ad}K_-)(\bar{P}_- + \mu_-)$$

with $\bar{P}_\pm \equiv \pm[\mu_\pm,\ P_\pm]$. Therefore, equations (25) become

$$\exp\int \mathrm{d}t\, \varphi(x_+, x_-, t)\tau(t) = \langle\tau|\mathrm{e}^{K_+} M_+ M_-^{-1} \mathrm{e}^{-K_-}|\tau\rangle$$

$$\int \mathrm{d}t\, \psi_-(x_+, x_-, t)\tau(t) = \frac{\langle\tau|\mathrm{e}^{K_+} M_+ M_-^{-1}[1 + \exp(-\mathrm{ad}K_-)P_-]\mathrm{e}^{-K_-}\int \mathrm{d}t\, e_-(t)|\tau\rangle}{\langle\tau|\mathrm{e}^{K_+} M_+ M_-^{-1}\mathrm{e}^{-K_-}|\tau\rangle}$$

$$\int \mathrm{d}t\, \psi_+(x_+, x_-, t)\tau(t) = \frac{\int \mathrm{d}t\langle\tau|e_+(t)\mathrm{e}^{K_+}[1 + \exp(-\mathrm{ad}K_+)P_+]M_+ M_-^{-1}\mathrm{e}^{-K_-}|\tau\rangle}{\langle\tau|\mathrm{e}^{K_+} M_+ M_-^{-1}\mathrm{e}^{-K_-}|\tau\rangle}.$$

Thus, provided we know enough about the highest-weight representations of the underlying continuum Lie algebra $\mathcal{G}$, we may be able to obtain the solution of the system (4), (5) using the above relations.

7. Concluding remarks

In this article, we have studied the three-dimensional two-extended Toda model by generalizing various techniques for treating two-dimensional models. The models studied here are usually integro-differential equations; there is, however, one special case, say, equations (7), which is a complete system of *differential* equations. Actually, this system is just the (B_2, C_2) flow of the so-called semiclassical or continuous Toda hierarchy proposed recently by Takasaki and Takebe [18]. The (B_1, C_1) flow of this hierarchy is the well known 'continuous Toda' model, which possesses the w_∞ symmetry and corresponds to real Euclidean Einstein spaces with at least one rotational Killing vector [5]. Therefore, it is of utmost interest to ask whether equations (7) correspond to any similar structure. If this is true, then it might also be interesting to clarify the roles of a more general system (5) in the gravitational theories.

Although we have given some hint of deriving formal solutions to the system (4), (5), we have to say that such constructions are really *formal*, since we do not have enough knowledge about the highest-weight representations of the continuum Lie algebras to determine whether our construction can really give rise to explicit, physically non-trivial and interesting solutions. Thus is seems necessary to continue the study of the structures and representations of the continuum Lie algebras themselves.

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